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Boundary Exchange Algebras and Scattering on the Half Line

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Abstract

Some algebraic aspects of field quantization in space-time with boundaries are discussed. We introduce an associative algebra $\mathcal{B}_R$, whose exchange properties are inferred from the scattering processes in integrable models with reflecting boundary conditions on the half line. The basic properties of $\mathcal{B}_R$ are established and the Fock representations associated with certain involutions in $\mathcal{B}_R$ are derived. We apply these results for the construction of quantum fields and for the study of scattering on the half line.

1. Introduction

It is well known that the presence of boundaries in space affects the behavior of quantum fields. In this paper we discuss the influence of the boundary conditions on the canonical commutation relations between creation and annihilation operators. Our investigation is inspired mainly by the factorized scattering theory of integrable models with reflecting boundary conditions on the half line. In the absence of boundaries [6,13,26], the algebraic features of these models are encoded in the Zamolodchikov-Faddeev (Z-F) algebra [6,26], denoted in what follows by $\mathcal{A}_R$. This is an associative algebra, whose generators satisfy quadratic constraints, known as exchange relations. The Fock representation of $\mathcal{A}_R$ equipped with an appropriate involution describes the scattering processes in integrable models. In this respect one should recall first that the Fock space contains two dense subspaces whose elements are interpreted as asymptotic in- and out-states. Second, the S-matrix can be explicitly constructed as a unitary operator interpolating between the asymptotic in- and out-spaces.

In a pioneering paper from the middle of the eighties, Cherednik [4] suggested a possible generalization of factorized scattering theory to integrable models with reflecting boundary conditions, which preserve integrability. The recent efforts to gain a deeper insight in various boundary-related two-dimensional phenomena, stimulated further investigations [5,7-12,16,21,23-25] in this subject. Among others, we would like to mention the attempts to develop an algebraic approach. One of the basic ideas there is to extend the Z-F algebra by introducing [8-12] "boundary creating" (also called "reflection") operators, which formally translate in algebraic terms the nontrivial boundary conditions. When possible, such an algebraic formulation is quite attractive because the treatment of the boundary conditions in their standard analytic form is as a rule a complicated matter. In spite of the great progress in implementing the above idea in particular models, the fundamental features of the boundary operators and their interplay with the "bulk" theory are still to be investigated. This is among the main purposes of the present paper.

We start our analysis by introducing an exchange algebra $\mathcal{B}_R$ with the following

structure. In the above spirit, $\mathcal{B}_R$ contains both boundary and bulk generators. The latter have a counterpart in $\mathcal{A}_R$, but we shall see that the exchange of two bulk generators of $\mathcal{B}_R$ involves in general boundary elements. The impact of the boundary on the bulk theory is therefore manifest already on the algebraic level, while the detailed boundary conditions are specified on the level of representation. We concentrate in this article on the Fock representations of $\mathcal{B}_R$. We will show that there exist two series of such representations, depending on certain involutions in $\mathcal{B}_R$. We shall construct these representations explicitly, establishing also their basic properties. As an application of these results, we will perform a detailed and rigorous investigation of the S-matrix of integrable models in the presence of reflecting boundaries.

The paper is organized as follows. In Sect. 2 we define the exchange algebra $\mathcal{B}_R$ and investigate some of its basic features. We introduce the concept of reflection $\mathcal{B}_R$-algebra and the related notion of reflection automorphism. At the end of this section we describe also a family of natural generalizations of $\mathcal{B}_R$. Sect. 3 is devoted to the Fock representations of $\mathcal{B}_R$. In Sect. 4 we describe some applications. We show that the second quantization on the half line naturally leads to $\mathcal{B}_R$. We also analyze here the scattering operator of integrable models. The last section contains our conclusions. In the appendix we construct representations of $\mathcal{B}_R$ carrying a boundary quantum number.

This article brings together and extends the results independently obtained by the present authors in [19] and [27].

2. The Exchange Algebra $\mathcal{B}_R$

$\mathcal{B}_R$ is by definition an associative algebra with identity element $\mathbf{1}$. It has two types of generators:

$$\{a_\alpha(x),\, a^{*\alpha}(x)\ :\ \alpha = 1, ..., N,\ x \in \mathbf{R}^s\} \tag{2.1}$$

and

$$\{b_\alpha^\beta(x)\ :\ \alpha, \beta = 1, ..., N,\ x \in \mathbf{R}^s\} \quad , \tag{2.2}$$

which, as mentioned in the introduction, are called bulk and boundary generators respectively. For convenience, we divide the constraints on (2.1,2) in three groups:

(i) bulk exchange relations are quadratic in the bulk generators and read

$$a_{\alpha_1}(x_1)\, a_{\alpha_2}(x_2) \;-\; R_{\alpha_2\alpha_1}^{\beta_1\beta_2}(x_2,x_1)\, a_{\beta_2}(x_2)\, a_{\beta_1}(x_1) \;=\; 0 \quad , \tag{2.3}$$

$$a^{*\alpha_1}(x_1)\, a^{*\alpha_2}(x_2) - a^{*\beta_2}(x_2)\, a^{*\beta_1}(x_1)\, R_{\beta_2\beta_1}^{\alpha_1\alpha_2}(x_2,x_1) = 0 \quad , \tag{2.4}$$

$$\begin{aligned} a_{\alpha_1}(x_1)\, a^{*\alpha_2}(x_2) \;-\; a^{*\beta_2}(x_2)\, R_{\alpha_1\beta_2}^{\alpha_2\beta_1}(x_1,x_2)\, a_{\beta_1}(x_1) = \\ \frac{1}{2}\,\delta(x_1-x_2)\,\delta_{\alpha_1}^{\alpha_2}\,\mathbf{1} + \frac{1}{2}\,\delta(x_1+x_2)\, b_{\alpha_1}^{\alpha_2}(x_1) \quad ; \end{aligned} \tag{2.5}$$

(ii) boundary exchange relations

$$\begin{aligned} R_{\alpha_1\alpha_2}^{\gamma_2\gamma_1}(x_1,x_2)\, b_{\gamma_1}^{\delta_1}(x_1)\, R_{\gamma_2\delta_1}^{\beta_1\delta_2}(x_2,-x_1)\, b_{\delta_2}^{\beta_2}(x_2) = \\ b_{\alpha_2}^{\gamma_2}(x_2)\, R_{\alpha_1\gamma_2}^{\delta_2\delta_1}(x_1,-x_2)\, b_{\delta_1}^{\gamma_1}(x_1)\, R_{\delta_2\gamma_1}^{\beta_1\beta_2}(-x_2,-x_1) \quad ; \end{aligned} \tag{2.6}$$

(iii) mixed relations

$$a_{\alpha_1}(x_1)\, b_{\alpha_2}^{\beta_2}(x_2) = R_{\alpha_2\alpha_1}^{\gamma_1\gamma_2}(x_2,x_1)\, b_{\gamma_2}^{\delta_2}(x_2)\, R_{\gamma_1\delta_2}^{\beta_2\delta_1}(x_1,-x_2)\, a_{\delta_1}(x_1) \quad , \tag{2.7}$$

$$b_{\alpha_1}^{\beta_1}(x_1)\, a^{*\alpha_2}(x_2) = a^{*\delta_2}(x_2)\, R_{\alpha_1\delta_2}^{\gamma_2\delta_1}(x_1,x_2)\, b_{\delta_1}^{\gamma_1}(x_1)\, R_{\gamma_2\gamma_1}^{\beta_1\alpha_2}(x_2,-x_1) \quad . \tag{2.8}$$

In the above equations and in what follows the summation over repeated upper and lower indices is always understood. The entries of the exchange factor R are complex valued measurable functions on $\mathbf{R}^s \times \mathbf{R}^s$, obeying

$$R_{\alpha_1\alpha_2}^{\gamma_1\gamma_2}(x_1,x_2)\, R_{\gamma_1\gamma_2}^{\beta_1\beta_2}(x_2,x_1) = \delta_{\alpha_1}^{\beta_1}\,\delta_{\alpha_2}^{\beta_2} \quad , \tag{2.9}$$

$$R_{\alpha_1\alpha_2}^{\gamma_1\gamma_2}(x_1,x_2)\, R_{\gamma_2\alpha_3}^{\delta_2\beta_3}(x_1,x_3)\, R_{\gamma_1\delta_2}^{\beta_1\beta_2}(x_2,x_3) = R_{\alpha_2\alpha_3}^{\gamma_2\gamma_3}(x_2,x_3)\, R_{\alpha_1\gamma_2}^{\beta_1\delta_2}(x_1,x_3)\, R_{\delta_2\gamma_3}^{\beta_2\beta_3}(x_1,x_2)\,. \tag{2.10}$$

These compatibility conditions are assumed throughout the paper and can be considered as general requirements on R, which together with eqs.(2.3-8) define the exchange algebra $\mathcal{B}_R$. Eq.(2.10) is the spectral quantum Yang-Baxter equation in its braid form, $\mathbf{R}^s$ playing the role of spectral set.

Let us comment now on the exchange relations (2.3-8), which may look at first sight a bit complicated. Concerning the general structure, we observe that after setting formally all boundary generators in (2.3-8) to zero and rescaling by a factor of $1/\sqrt{2}$ the bulk generators, one gets the Z-F algebra $\mathcal{A}_R$. This fact clarifies partially the origin of eqs.(2.3-5). The presence of boundary generators in the right hand side of (2.5) is worth stressing. This is one of the essential points, in which our approach differs from the previous attempts to define a boundary exchange algebra.

Eq.(2.6) describes the exchange of two boundary generators taken in generic points and also deserves a remark. It looks similar to the boundary Yang-Baxter equation [4]; the difference is that the elements $\{b_\alpha^\beta(x)\}$ do not commute in general and consequently their position in (2.6) is essential. Notice also that $\{b_\alpha^\beta(x)\}$ close a subalgebra of $\mathcal{B}_R$, which presents by itself some interest [24]. Finally, eqs.(2.7,8) express the interplay between $\{a_\alpha(x),\, a^{*\alpha}(x)\}$ and $\{b_\alpha^\beta(x)\}$ and represent another relevant new aspect of our proposal.

Two straightforward examples, denoted by $\mathcal{B}_\pm$, correspond to the constant solutions

$$R_{\alpha_1\alpha_2}^{\beta_1\beta_2} = \pm\, \delta_{\alpha_1}^{\beta_2}\, \delta_{\alpha_2}^{\beta_1} \tag{2.11}$$

of (2.9,10) and represent in the above context the counterparts of the canonical (anti)commutation relations. Eqs.(2.6-8) imply that $\{b_\alpha^\beta(x)\}$ are central elements in $\mathcal{B}_\pm$. Nevertheless, also in these relatively simple cases the right hand side of eq.(2.5) keeps trace of the nontrivial boundary conditions. Two applications of $\mathcal{B}_+$ with $N=1$ are described in Sect. 4.

To further understand the structure of $\mathcal{B}_R$ and its representations, it is instructive to introduce some involutions in $\mathcal{B}_R$. Let H_N be the family of invertible Hermitian $N\times N$ matrices and let $\mathcal{M}$ be the set of matrix valued functions $m\,:\,\mathbf{R}^s \to H_N$, such that the entries of $m(x)$ and $m(x)^{-1}$ are measurable and bounded in $\mathbf{R}^s$. Consider the mapping I_m defined by

$$I_m\;:\; a^{*\alpha}(x) \longmapsto m_\alpha^\beta(x)\, a_\beta(x) \quad , \tag{2.12}$$

$$I_m\;:\; a_\alpha(x) \longmapsto a^{*\beta}(x)\, {m^{-1}}_\beta^\alpha(x) \quad , \tag{2.13}$$

$$I_m \,:\, b_\alpha^\beta(x) \longmapsto m_\beta^\gamma(-x)\, b_\gamma^\delta(-x)\, m^{-1}{}_\delta^\alpha(x) \quad . \tag{2.14}$$

Provided that $m \in \mathcal{M}$ satisfies

$$R^{\dagger\gamma_1\gamma_2}_{\alpha_1\alpha_2}(x_1,x_2)\, m^{\beta_1}_{\gamma_1}(x_1)\, m^{\beta_2}_{\gamma_2}(x_2) = m^{\gamma_1}_{\alpha_1}(x_2)\, m^{\gamma_2}_{\alpha_2}(x_1)\, R^{\beta_1\beta_2}_{\gamma_1\gamma_2}(x_2,x_1) \quad , \tag{2.15}$$

it is not difficult to check that when extended as an antilinear antihomomorphism on $\mathcal{B}_R$, I_m defines an involution. In eq.(2.15) and in what follows the dagger stands for Hermitian conjugation, i.e.

$$R^{\dagger\beta_1\beta_2}_{\alpha_1\alpha_2}(x_1,x_2) \equiv \overline{R}^{\alpha_1\alpha_2}_{\beta_1\beta_2}(x_1,x_2) \quad ,$$

the bar indicating complex conjugation. Notice that for the algebras $\mathcal{B}_\pm$ eq.(2.15) is satisfied for any $m \in \mathcal{M}$.

In this paper we shall concentrate on the following specific type of $\mathcal{B}_R$-algebras. We call the boundary generators $\{b_\alpha^\beta(x)\}$ reflections if

$$b_\alpha^\gamma(x)\, b_\gamma^\beta(-x) = \delta_\alpha^\beta \tag{2.16}$$

hold. In this case we refer to $\mathcal{B}_R$ as reflection exchange algebra. The condition (2.16) is I_m-invariant and one easily proves

Proposition 1. *Let $\mathcal{B}_R$ be a reflection exchange algebra. Then the mapping*

$$\varrho \,:\, a_\alpha(x) \longmapsto b_\alpha^\beta(x) a_\beta(-x) \quad , \tag{2.17}$$

$$\varrho \,:\, a^{*\alpha}(x) \longmapsto a^{*\beta}(-x) b_\beta^\alpha(-x) \quad , \tag{2.18}$$

$$\varrho \,:\, b_\alpha^\beta(x) \longmapsto b_\alpha^\beta(x) \quad , \tag{2.19}$$

leaves invariant the constraints (2.3-8) *and extends therefore to an automorphism on $\mathcal{B}_R$. Moreover, being compatible with I_m, ϱ is actually an automorphism of $\{\mathcal{B}_R, I_m\}$ considered as an algebra with involution.*

In what follows ϱ is called the reflection automorphism of $\mathcal{B}_R$. Besides encoding some essential features of any reflection exchange algebra, ϱ has a direct physical

interpretation in scattering theory: it provides a mathematical description of the intuitive physical picture that bouncing back from a wall, particles change the sign of their rapidities. In fact, the two elements $a^{*\alpha}(-x)$ and $a^{*\beta}(x)b_\beta^\alpha(x)$ are ϱ-equivalent,

$$a^{*\alpha}(-x) \sim a^{*\beta}(x)b_\beta^\alpha(x) \quad . \tag{2.20}$$

This relation in our framework is the counterpart of a heuristic equation (see for example eq.(3.22) of [10]), conjectured in all papers dealing with factorized scattering with reflecting boundaries. In the next section we will show that the ϱ-equivalence becomes actually an equality in the Fock representation of $\{\mathcal{B}_R, I_m\}$. For proving this statement we will use the relations

$$\{a_\alpha(x_1) - \varrho\,[a_\alpha(x_1)]\}\, a^{*\beta}(x_2) = a^{*\gamma}(x_2)\, R_{\alpha\gamma}^{\beta\delta}(x_1,x_2)\, \{a_\delta(x_1) - \varrho\,[a_\delta(x_1)]\} \quad , \tag{2.21}$$

$$\{a^{*\alpha}(x_1) - \varrho\,[a^{*\alpha}(x_1)]\}\, a^{*\beta}(x_2) = a^{*\gamma}(x_2)\, \left\{a^{*\delta}(x_1) - \varrho\,\left[a^{*\delta}(x_1)\right]\right\}\, R_{\gamma\delta}^{\alpha\beta}(x_2,x_1) \quad , \tag{2.22}$$

whose validity follows directly from eqs.(2.3-5,7,8,16).

Before concluding this section, we would like to introduce a whole class of more general exchange algebras which can be treated in the above way. The idea is to replace the reflection $x \mapsto -x$, which plays a special role in defining $\mathcal{B}_R$, with any almost everywhere differentiable mapping $\lambda\,:\, x \mapsto \widetilde{x}$, satisfying the iterative functional equation

$$\lambda\,(\lambda(x)) = x \quad . \tag{2.23}$$

The resulting exchange algebras will be denoted by $\mathcal{B}_{R,\lambda}$ and are characterized by the following constraints: the relations (2.3,4) remain unchanged, whereas (2.5-8) take the form

$$\begin{aligned} &a_{\alpha_1}(x_1)\, a^{*\alpha_2}(x_2) \;-\; a^{*\beta_2}(x_2)\, R_{\alpha_1\beta_2}^{\alpha_2\beta_1}(x_1,x_2)\, a_{\beta_1}(x_1) = \\ &\frac{1}{2|\mathrm{det}\lambda'(x_1)|^{1/2}}\, \left\{\delta(x_1-x_2)\, \delta_{\alpha_1}^{\alpha_2}\, \mathbf{1} + \delta(x_1-\widetilde{x}_2)\, b_{\alpha_1}^{\alpha_2}(x_1)\right\} \quad , \end{aligned} \tag{2.24}$$

$$\begin{aligned} &R_{\alpha_1\alpha_2}^{\gamma_2\gamma_1}(x_1,x_2)\, b_{\gamma_1}^{\delta_1}(x_1)\, R_{\gamma_2\delta_1}^{\beta_1\delta_2}(x_2,\widetilde{x}_1)\, b_{\delta_2}^{\beta_2}(x_2) = \\ &b_{\alpha_2}^{\gamma_2}(x_2)\, R_{\alpha_1\gamma_2}^{\delta_2\delta_1}(x_1,\widetilde{x}_2)\, b_{\delta_1}^{\gamma_1}(x_1)\, R_{\delta_2\gamma_1}^{\beta_1\beta_2}(\widetilde{x}_2,\widetilde{x}_1) \quad , \end{aligned} \tag{2.25}$$

$$
\begin{aligned}
&a_{\alpha_1}(x_1)\, b^{\beta_2}_{\alpha_2}(x_2) = R^{\gamma_1\gamma_2}_{\alpha_2\alpha_1}(x_2,x_1)\, b^{\delta_2}_{\gamma_2}(x_2)\, R^{\beta_2\delta_1}_{\gamma_1\delta_2}(x_1,\widetilde{x}_2)\, a_{\delta_1}(x_1) \quad , \\
&b^{\beta_1}_{\alpha_1}(x_1)\, a^{*\alpha_2}(x_2) = a^{*\delta_2}(x_2)\, R^{\gamma_2\delta_1}_{\alpha_1\delta_2}(x_1,x_2)\, b^{\gamma_1}_{\delta_1}(x_1)\, R^{\beta_1\alpha_2}_{\gamma_2\gamma_1}(x_2,\widetilde{x}_1) \quad .
\end{aligned}
\tag{2.26}
$$

Here $\lambda'(x)$ denotes the Jacobian matrix of the function λ. The results of this section regarding $\mathcal{B}_R$ can be transferred with obvious modifications to $\mathcal{B}_{R,\lambda}$. For the complete set of solutions of eq.(2.23) we refer to [14]. When $s=1$ for instance, the mapping λ can be any almost everywhere differentiable function in $\mathbf{R}$ whose graph is symmetric with respect to the diagonal $\{(x,y)\in \mathbf{R}^2 \,:\, x=y\}$.

Summarizing, we introduced so far the exchange algebra $\mathcal{B}_R$ and some natural generalizations of it. We defined also a set of involutions in $\mathcal{B}_R$, which are useful in representation theory. Focusing on reflection type $\mathcal{B}_R$-algebras, we shall construct in the next section the relative Fock representations.

3. Fock Representations

We consider in this paper representations of $\{\mathcal{B}_R, I_m\}$ with the following general structure.

1. The representation space $\mathcal{L}$ is a locally convex and complete topological linear space over $\mathbf{C}$.
2. The generators $\{a_\alpha(x),\, a^{*\alpha}(x),\, b^\beta_\alpha(x)\}$ are operator valued distributions with common and invariant dense domain $\mathcal{D}\subset\mathcal{L}$, where eqs.(2.3-8) hold.
3. $\mathcal{D}$ is equipped with a nondegenerate sesquilinear form (inner product) $\langle\cdot\,,\,\cdot\rangle_m$, which is at least separately continuous. The involution I_m defined by eqs.(2.12-14) is realized as a conjugation with respect to $\langle\cdot\,,\,\cdot\rangle_m$.

A Fock representation of $\{\mathcal{B}_R, I_m\}$ is specified further by the following requirement.

4. There exists a vector (vacuum state) $\Omega\in\mathcal{D}$ which is annihilated by $a_\alpha(x)$. Moreover, Ω is cyclic with respect to $\{a^{*\alpha}(x)\}$ and $\langle\Omega\,,\,\Omega\rangle_m = 1$.

A more general situation, when a boundary quantum number [10] is present, is outlined in the appendix.

There is a series of direct but quite important corollaries from the above assumptions. Let us start with

Proposition 2. *The automorphism ϱ of any reflection $\mathcal{B}_R$-algebra is implemented in the above Fock representations by the identity operator.*

Proof. First of all we observe that

$$\langle P'[a^*]\Omega\,,\,\{a_\alpha(x) - \varrho[a_\alpha(x)]\}P[a^*]\Omega\rangle_m = 0 \quad , \tag{3.1}$$

where P and P' are arbitrary polynomials. In fact, by means of eq.(2.21) one can shift the curly bracket to the vacuum and use that $a_\alpha(x)$ annihilate Ω. Now the cyclicity of Ω, combined with the properties of $\langle\,\cdot\,,\,\cdot\,\rangle_m$, allow to replace $P'[a^*]\Omega$ by an arbitrary state $\varphi \in \mathcal{D}$. A further conjugation leads to

$$\langle P[a^*]\Omega\,,\,\{a^{*\alpha}(x) - \varrho[a^{*\alpha}(x)]\}\varphi\rangle_m = 0 \quad , \tag{3.2}$$

which implies

$$a^{*\alpha}(x) = a^{*\beta}(-x)b_\beta^\alpha(-x) \tag{3.3}$$

on $\mathcal{D}$. Analogously, employing (2.22) one concludes that

$$a_\alpha(x) = b_\alpha^\beta(x)a_\beta(-x) \tag{3.4}$$

also holds on $\mathcal{D}$. Finally, taking in consideration eq.(2.19) we deduce that ϱ is indeed implemented by the identity operator.

For describing some further characteristic features of the Fock representations of $\mathcal{B}_R$, we introduce the c-number distributions

$$B_\alpha^\beta(x) \equiv \langle\Omega\,,\,b_\alpha^\beta(x)\Omega\rangle_m \quad . \tag{3.5}$$

The requirement 3 implies that

$$B^{\dagger}{}_\alpha^\beta(x) = m_\alpha^\gamma(-x)\,B_\gamma^\delta(-x)\,m^{-1}{}_\delta^\beta(x) \quad , \tag{3.6}$$

which is the analog of condition (2.15) regarding the exchange factor R.

Two other simple consequences of our assumptions 1-4 above are collected in

Proposition 3. *The vacuum vector* Ω *is unique* (*up to a phase factor*) *and satisfies*

$$b_\alpha^\beta(x)\,\Omega = B_\alpha^\beta(x)\,\Omega \quad . \tag{3.7}$$

Proof. The argument implying the uniqueness of the vacuum is standard. Concerning eq.(3.7), it can be inferred from the identity

$$\langle [b_\alpha^\beta(x) - B_\alpha^\beta(x)]\Omega\,,\, P[a^*]\Omega\rangle_m = 0 \quad , \tag{3.8}$$

P being an arbitrary polynomial. In order to prove eq.(3.8) we shift by a conjugation the polynomial to the first factor in the right hand side of (3.8) and apply afterwards the exchange relation (2.8) and eq.(3.5). For completing the proof, one also employs that Ω is cyclic and $\langle\,\cdot\,,\,\cdot\,\rangle_m$ is continuous and nondegenerate.

Combining eq.(3.7) with the fact that $a_\alpha(x)$ annihilate Ω, we conclude that eqs.(2.5,7,8) allow for a purely algebraic derivation of the vacuum expectation values involving any number and combination of the generators $\{a_\alpha(x),\, a^{*\alpha}(x),\, b_\alpha^\beta(x)\}$. In particular, taking the vacuum expectation value of eq.(2.6) one gets

$$\begin{gathered} R_{\alpha_1\alpha_2}^{\gamma_2\gamma_1}(x_1,x_2)\, B_{\gamma_1}^{\delta_1}(x_1)\, R_{\gamma_2\delta_1}^{\beta_1\delta_2}(x_2,-x_1)\, B_{\delta_2}^{\beta_2}(x_2) = \\ B_{\alpha_2}^{\gamma_2}(x_2)\, R_{\alpha_1\gamma_2}^{\delta_2\delta_1}(x_1,-x_2)\, B_{\delta_1}^{\gamma_1}(x_1)\, R_{\delta_2\gamma_1}^{\beta_1\beta_2}(-x_2,-x_1) \quad . \end{gathered} \tag{3.9}$$

We thus recover at the level of Fock representation the original boundary Yang-Baxter equation [4]. In addition, when one is dealing with reflection algebras, eq.(2.17) implies

$$B_\alpha^\gamma(x)\, B_\gamma^\beta(-x) = \delta_\alpha^\beta \quad . \tag{3.10}$$

In this case we refer to B as reflection matrix.

A final comment in this introductory part concerns the algebras $\mathcal{B}_\pm$. Using that $\{b_\alpha^\beta(x)\}$ are central elements, in a Fock representation of $\mathcal{B}_\pm$ one has

$$b_\alpha^\beta(x)\varphi = B_\alpha^\beta(x)\varphi \tag{3.11}$$

for any $\varphi \in \mathcal{D}$.

At this stage it is convenient to introduce the set $\mathcal{M}(R,B)$ of all elements of $\mathcal{M}$ obeying both eqs.(2.15) and (3.6). Then the basic input fixing a Fock representation of the reflection algebra $\{\mathcal{B}_R, I_m\}$ is the triplet $\{R,\, B;\, m\}$, where R and B satisfy eqs.(2.9,10) and (3.9,10), and $m \in \mathcal{M}(R,B)$. Some explicit examples of such triplets have been found already by Cherednik [4]. With any $\{R,\, B;\, m\}$ we associate a Fock representation denoted by $\mathcal{F}_{R,B;m}$. To the end of this section we will describe the explicit construction of $\mathcal{F}_{R,B;m}$.

Our first step will be to introduce the n-particle subspace $\mathcal{H}^n_{R,B}$ of $\mathcal{F}_{R,B;m}$. For this purpose we consider

$$\mathcal{H} = \bigoplus_{\alpha=1}^{N} L^2(\mathbf{R}^s) \quad , \tag{3.12}$$

equipped with the standard scalar product

$$(\varphi, \psi) = \int d^s x \varphi^{\dagger\alpha}(x)\psi_\alpha(x) = \sum_{\alpha=1}^{N} \int d^s x \overline{\varphi}_\alpha(x)\psi_\alpha(x) \quad . \tag{3.13}$$

For $n \geq 1$ the n-particle space $\mathcal{H}^n_{R,B}$ we are looking for, will be a subspace of the n-fold tensor power $\mathcal{H}^{\otimes n}$, characterized by a suitable projection operator $P^{(n)}_{R,B}$. The ingredients for constructing $P^{(n)}_{R,B}$ are essentially two: a specific finite group and its representation in $\mathcal{H}^{\otimes n}$, defined in terms of the exchange factor R and the reflection matrix B.

Let us concentrate first on the group. In the case of $\mathcal{A}_R$, this was [17] simply the permutation group $\mathcal{P}_n$. The physics behind $\mathcal{B}_R$ suggest to enlarge in this case the group by adding a reflection generator. More precisely, we consider the group $\mathcal{W}_n$ generated by $\{\tau,\, \sigma_i \ :\ i = 1, ..., n-1\}$ which satisfy

$$\begin{aligned} &\sigma_i\, \sigma_j = \sigma_j\, \sigma_i \quad , \qquad |i-j| \geq 2 \quad , \\ &\sigma_i\, \tau = \tau\, \sigma_i \quad , \qquad 1 \leq i < n-2 \quad , \end{aligned} \tag{3.14}$$

$$\begin{aligned} \sigma_i\, \sigma_{i+1}\, \sigma_i &= \sigma_{i+1}\, \sigma_i\, \sigma_{i+1} \quad , \\ \sigma_{n-1}\, \tau\, \sigma_{n-1}\, \tau &= \tau\, \sigma_{n-1}\, \tau\, \sigma_{n-1} \quad , \end{aligned} \tag{3.15}$$

$$\sigma_i^2 = \tau^2 = \mathbf{1} \quad . \tag{3.16}$$

$\mathcal{W}_n$ is the Weyl group associated with the root systems of the classical Lie algebra B_n and has $2^n n!$ elements. Although it contains no permutations, $\mathcal{W}_1 = \{\mathbf{1}, \tau\}$ is nontrivial.

We turn now to the representation of $\mathcal{W}_n$ in $\mathcal{H}^{\otimes n}$. Observing that any element $\varphi \in \mathcal{H}^{\otimes n}$ can be viewed as a column whose entries are $\varphi_{\alpha_1 \cdots \alpha_n}(x_1, \ldots, x_n)$, we define the operators $\{T^{(n)},\, S_i^{(n)} \;:\; i = 1, ..., n-1\}$ acting on $\mathcal{H}^{\otimes n}$ according to:

$$\left[S_i^{(n)} \varphi\right]_{\alpha_1 \ldots \alpha_n} (x_1, ..., x_i, x_{i+1}, ..., x_n) =$$

$$[R_{i\, i+1}(x_i, x_{i+1})]_{\alpha_1 \ldots \alpha_n}^{\beta_1 \ldots \beta_n}\, \varphi_{\beta_1 \ldots \beta_n}(x_1, ..., x_{i+1}, x_i, ..., x_n) \quad , \qquad n \geq 2 \quad , \tag{3.17}$$

$$\left[T^{(n)} \varphi\right]_{\alpha_1 \ldots \alpha_n} (x_1, ..., x_n) =$$

$$[B_n(x_n)]_{\alpha_1 \ldots \alpha_n}^{\beta_1 \ldots \beta_n}\, \varphi_{\beta_1 \ldots \beta_n}(x_1, ..., x_{n-1}, -x_n) \quad , \qquad n \geq 1 \tag{3.18}$$

where

$$[R_{ij}(x_i, x_j)]_{\alpha_1 \ldots \alpha_n}^{\beta_1 \ldots \beta_n} = \delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2} \cdots \widehat{\delta_{\alpha_i}^{\beta_i}} \cdots \widehat{\delta_{\alpha_j}^{\beta_j}} \cdots \delta_{\alpha_n}^{\beta_n}\, R_{\alpha_i \alpha_j}^{\beta_i \beta_j}(x_i, x_j) \tag{3.19}$$

and

$$[B_i(x)]_{\alpha_1 \ldots \alpha_n}^{\beta_1 \ldots \beta_n} = \delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2} \cdots \widehat{\delta_{\alpha_i}^{\beta_i}} \cdots \delta_{\alpha_n}^{\beta_n}\, B_{\alpha_i}^{\beta_i}(x) \quad . \tag{3.20}$$

The hat in eqs.(3.19,20) indicates that the corresponding symbol must be omitted. For implementing eqs.(3.17,18) on the whole $\mathcal{H}^{\otimes n}$, we assume at this stage that the matrix elements $R_{\alpha_1 \alpha_2}^{\beta_1 \beta_2}(x_1, x_2)$ and $B_\alpha^\beta(x)$ are bounded functions. We are now in position to prove

Proposition 4: $\{T^{(n)},\, S_i^{(n)} \;:\; i = 1, ..., n-1\}$ *are bounded operators on* $\mathcal{H}^{\otimes n}$ *and the mapping*

$$\chi^{(n)} \;:\; \tau \longmapsto T^{(n)} \quad , \qquad \chi^{(n)} \;:\; \sigma_i \longmapsto S_i^{(n)} \quad , \qquad i = 1, \cdots, n-1 \tag{3.21}$$

defines a representation of $\mathcal{W}_n$ *in* $\mathcal{H}^{\otimes n}$. *Moreover,*

$$P_{R,B}^{(n)} \equiv \frac{1}{2^n n!} \sum_{\nu \in \mathcal{W}_n} \chi^{(n)}(\nu) \tag{3.22}$$

is a bounded projection operator in $\mathcal{H}^{\otimes n}$.

Proof. The main point is to show that $\{T^{(n)},\, S_i^{(n)} \; : \; i = 1, ..., n-1\}$ obey eqs.(3.14-16). This can be checked directly. Eqs.(3.14) are satisfied by construction. Eqs.(3.15) follow from (2.10) and (3.9). Finally, eqs.(2.9) and (3.10) imply (3.16).

Let us observe in passing that $P_{R,B}^{(n)}$ is an orthogonal projector only if the $N \times N$ identity matrix e belongs to $\mathcal{M}(R, B)$. In general $P_{R,B}^{(n)}$ is not orthogonal, but being a bounded operator determines for any $n \geq 1$ a (nonempty) closed subspace

$$\mathcal{H}_{R,B}^n \equiv P_{R,B}^{(n)} \mathcal{H}^{\otimes n} \quad . \tag{3.23}$$

By construction the elements of $\mathcal{H}_{R,B}^n$ behave as follows:

$$\varphi_{\alpha_1 \ldots \alpha_n}(x_1, ..., x_i, x_{i+1}, ..., x_n) = [R_{i\, i+1}(x_i, x_{i+1})]_{\alpha_1 \ldots \alpha_n}^{\beta_1 \ldots \beta_n} \varphi_{\beta_1 \ldots \beta_n}(x_1, ..., x_{i+1}, x_i, ..., x_n) \ , \tag{3.24}$$

$$\varphi_{\alpha_1 \ldots \alpha_n}(x_1, ..., x_n) = [B_n(x_n)]_{\alpha_1 \ldots \alpha_n}^{\beta_1 \ldots \beta_n} \varphi_{\beta_1 \ldots \beta_n}(x_1, ..., x_{n-1}, -x_n) \quad . \tag{3.25}$$

Setting $\mathcal{H}_{R,B}^0 = \mathbf{C}^1$, we introduce also the finite particle space $\mathcal{F}_{R,B;m}^0(\mathcal{H})$ as the (complex) linear space of sequences $\varphi = \left(\varphi^{(0)}, \varphi^{(1)}, ..., \varphi^{(n)}, ...\right)$ with $\varphi^{(n)} \in \mathcal{H}_{R,B}^n$ and $\varphi^{(n)} = 0$ for n large enough. The vacuum state is $\Omega = (1, 0, ..., 0, ...)$.

At this point we define on $\mathcal{F}_{R,B;m}^0(\mathcal{H})$ the annihilation and creation operators $\{a(f), a^*(f) \, : \, f \in \mathcal{H}\}$ setting $a(f)\Omega = 0$ and

$$[a(f)\varphi]_{\alpha_1 \cdots \alpha_n}^{(n)}(x_1, ..., x_n) = \sqrt{n+1} \int d^s x \, f^{\dagger \alpha_0}(x) \varphi_{\alpha_0 \alpha_1 \cdots \alpha_n}^{(n+1)}(x, x_1, ..., x_n) \quad , \tag{3.26}$$

$$[a^*(f)\varphi]_{\alpha_1 \cdots \alpha_n}^{(n)}(x_1, ..., x_n) = \sqrt{n} \left[P_{R,B}^{(n)} f \otimes \varphi^{(n-1)}\right]_{\alpha_1 \cdots \alpha_n}(x_1, ..., x_n) \quad , \tag{3.27}$$

for all $\varphi \in \mathcal{F}_{R,B;m}^0(\mathcal{H})$. The operators $a(f)$ and $a^*(f)$ are in general unbounded on $\mathcal{F}_{R,B;m}^0(\mathcal{H})$. However, for any $\psi^{(n)} \in \mathcal{H}_{R,B}^n$ one has the estimates

$$\| \, a(f)\psi^{(n)} \, \| \leq \sqrt{n} \, \| \, f \, \| \| \, \psi^{(n)} \, \| \ , \qquad \| \, a^*(f)\psi^{(n)} \, \| \leq \sqrt{n} \, \| \, P_{R,B}^{(n+1)} \, \| \| \, f \, \| \| \, \psi^{(n)} \, \| \ , \tag{3.28}$$

$\| \cdot \|$ being the L^2-norm. Therefore $a(f)$ and $a^*(f)$ are bounded on each $\mathcal{H}_{R,B}^n$.

The right hand side of eq.(3.27) can be given an alternative form by implementing explicitly the action of $P_{R,B}^{(n)}$. The resulting expression is a bit complicated, but since in some cases it might be instructive, we give it for completeness:

$$[a^*(f)\varphi]_{\alpha_1 \cdots \alpha_n}^{(n)}(x_1, ..., x_n) = \frac{1}{2\sqrt{n}} \left[f_{\alpha_1}(x_1) \varphi_{\alpha_2 \cdots \alpha_n}^{(n-1)}(x_2, \ldots, x_n) + \right.$$

$$C(x_1;x_2,...,x_n)^{\beta_1\cdots\beta_n}_{\alpha_1\cdots\alpha_n} f_{\beta_1}(-x_1)\varphi^{(n-1)}_{\beta_2\cdots\beta_n}(x_2,\ldots,x_n)]+$$

$$\frac{1}{2\sqrt{n}}\sum_{k=2}^{n}[R_{k-1\,k}(x_{k-1},x_k)\cdots R_{1\,2}(x_1,x_k)]^{\beta_1\cdots\beta_n}_{\alpha_1\cdots\alpha_n}\left[f_{\beta_1}(x_k)\varphi^{(n-1)}_{\beta_2\cdots\beta_n}(x_1,\ldots,\widehat{x}_k,\ldots,x_n)+\right.$$

$$\left. C(x_k;x_1,...,\widehat{x}_k,...,x_n)^{\gamma_1\cdots\gamma_n}_{\beta_1\cdots\beta_n} f_{\gamma_1}(-x_k)\varphi^{(n-1)}_{\gamma_2\cdots\gamma_n}(x_1,\ldots,\widehat{x}_k,\ldots,x_n)\right] \quad , \tag{3.29}$$

where

$$C(x_k;x_1,...,\widehat{x}_k,...,x_n)^{\beta_1\cdots\beta_n}_{\alpha_1\cdots\alpha_n} =$$

$$\big[R_{12}(x_k,x_1)R_{23}(x_k,x_2)\cdots\widehat{R}_{k\,(k+1)}(x_k,x_k)\cdots R_{(n-1)\,n}(x_k,x_n)B_n(x_k)\cdot$$

$$R_{(n-1)\,n}(x_n,-x_k)\cdots\widehat{R}_{k\,(k+1)}(x_k,-x_k)\cdots R_{23}(x_2,-x_k)R_{12}(x_1,-x_k)\big]^{\beta_1\cdots\beta_n}_{\alpha_1\cdots\alpha_n} . \tag{3.30}$$

We turn now to the boundary generators, defining $b^{\beta}_{\alpha}(x)$ as the multiplicative operator whose action on $\mathcal{F}^0_{R,B;m}(\mathcal{H})$ is given by eq.(3.7) and

$$\left[b^{\beta}_{\alpha}(x)\varphi\right]^{(n)}_{\gamma_1\ldots\gamma_n}(x_1,...,x_n) = [R_{01}(x,x_1)\,R_{12}(x,x_2)\cdots R_{(n-1)\,n}(x,x_n)\,B_n(x)\cdot$$

$$\cdot R_{(n-1)\,n}(x_n,-x)\cdots R_{12}(x_2,-x)\,R_{01}(x_1,-x)]^{\beta\delta_1\ldots\delta_n}_{\alpha\gamma_1\ldots\gamma_n}\varphi^{(n)}_{\delta_1\ldots\delta_n}(x_1,...,x_n) \quad , \tag{3.31}$$

for $n\geq 1$. Notice that the boundary generators $\{b^{\beta}_{\alpha}(x)\}$ preserve the particle number.

By construction $\{a(f),\, a^*(f)\}$ and $\{b^{\beta}_{\alpha}(x)\}$ leave invariant $\mathcal{F}^0_{R,B;m}(\mathcal{H})$, which we take as the domain $\mathcal{D}$, whose existence was required in the definition of Fock representation. For deriving the commutation properties on $\mathcal{D}$ it is convenient to introduce the operator-valued distributions $a_{\alpha}(x)$ and $a^{*\alpha}(x)$ defined by

$$a(f) = \int d^sx\, f^{\dagger\alpha}(x)a_{\alpha}(x) \quad , \qquad a^*(f) = \int d^sx\, f_{\alpha}(x)a^{*\alpha}(x) \quad . \tag{3.32}$$

After a straightforward but lengthly computation, one verifies the validity of the following statement.

Proposition 5. *The operator-valued distributions* $\{a_{\alpha}(x), a^{*\alpha}(x)\}$ *and* $\{b^{\beta}_{\alpha}(x)\}$ *satisfy the relations* (2.3-8) *on* $\mathcal{D}$.

Assuming that $\mathcal{M}(R,B)\neq\emptyset$, we proceed further by implementing the involutions $\{I_m\ :\ m\in\mathcal{M}(R,B)\}$. For this purpose we have to construct a sesquilinear form

$\langle\cdot\,,\cdot\rangle_m$ on $\mathcal{D}$, such that the mapping (2.12-14) is realized as the conjugation with respect $\langle\cdot\,,\cdot\rangle_m$. Let us consider the following form on $\mathcal{D}$:

$$\langle\varphi,\psi\rangle_m = \sum_{n=0}^{\infty}\langle\varphi^{(n)},\psi^{(n)}\rangle_m \quad , \tag{3.33}$$

where

$$\langle\varphi^{(0)},\psi^{(0)}\rangle_m = \overline{\varphi}^{(0)}\psi^{(0)} \quad , \tag{3.34}$$

$$\langle\varphi^{(n)},\psi^{(n)}\rangle_m =$$

$$\int dx_1\cdots dx_n\varphi^{(n)\dagger\alpha_1\ldots\alpha_n}(x_1,...,x_n)m_{\alpha_1}^{\beta_1}(x_1)\cdots m_{\alpha_n}^{\beta_n}(x_n)\psi_{\beta_1\ldots\beta_n}^{(n)}(x_1,...,x_n) \quad . \tag{3.35}$$

The right hand side of (3.33) always makes sense because for any $\varphi,\psi\in\mathcal{D}$ the series is actually a finite sum. Using that $m(x)$ satisfies eqs.(2.15) and (3.6), one easily proves

Proposition 6. *The inner product defined by* (3.33-35) *is nondegenerate on* $\mathcal{D}$ *and the involution* I_m *is implemented by* $\langle\cdot\,,\cdot\rangle_m$*-conjugation.*

The next question concerns the positivity of $\langle\cdot\,,\cdot\rangle_m$. This point is conveniently discussed after introducing the subset $\mathcal{M}(R,B)_+$ of those elements of $\mathcal{M}(R,B)$, which are positive definite almost everywhere in $\mathbf{R}^s$. One has indeed

Proposition 7. *The inner product* $\langle\cdot\,,\cdot\rangle_m$ *is positive definite on* $\mathcal{D}$ *if and only if* $m\in\mathcal{M}(R,B)_+$.

Proof. From eq.(3.35) it is clear that if $m\in\mathcal{M}(R,B)_+$ then the inner product is positive definite. Conversely, suppose that $\langle\cdot\,,\cdot\rangle_m$ is positive definite. Let $y\in\mathbf{R}^s$ be a fixed non zero vector, and take an arbitrary $f\in\mathcal{H}$ with support laying in the half space $x\cdot y\geq 0$. Consider the 1-particle state

$$\varphi_\alpha(x) = [P_{R,B}^{(1)}f]_\alpha(x) = \frac{1}{2}\left[f_\alpha(x) + B_\alpha^\beta(x)f_\beta(-x)\right] \quad . \tag{3.36}$$

Using eqs. (3.6,10) and the support properties of f_α, one gets

$$\langle\varphi\,,\varphi\rangle_m = \frac{1}{2}\int d^sx\, f^{\dagger\alpha}(x)m_\alpha^\beta(x)f_\beta(x) \quad . \tag{3.37}$$

Since f is arbitrary, positivity of $\langle\,\cdot\,,\,\cdot\,\rangle_m$ implies that $m(x)$ is positive definite almost everywhere in the half space $x \cdot y \geq 0$. Finally, the arbitrariness of y allows to extend the validity of this conclusion to $\mathbf{R}^s$.

Proposition 7 shows that there are two kinds of Fock representations of $\mathcal{B}_R$. The representation $\mathcal{F}_{R,B;m}$ will be called of type A if $\langle\,\cdot\,,\,\cdot\,\rangle_m$ is positive definite; otherwise we will say that $\mathcal{F}_{R,B;m}$ is of type B. The standard probabilistic interpretation of quantum field theory applies directly only to the A-series. This does not mean however that the B-series has no physical applications. In the last case one has to isolate first a physical subspace where $\langle\,\cdot\,,\,\cdot\,\rangle_m$ is nonnegative. This is usually done by symmetry consideration and may depend on the specific model under consideration.

The final step in completing the derivation of $\mathcal{F}_{R,B;m}$ is the construction of the representation space $\mathcal{L}$. It is necessary at this stage to consider the classes A and B separately. For $m \in \mathcal{M}(R,B)_+$ the inner product space $\{\mathcal{D}, \langle\,\cdot\,,\,\cdot\,\rangle_m\}$ is actually a pre-Hilbert space. Let $\mathcal{F}_{R,B;m}(\mathcal{H})$ be the completion of $\mathcal{D}$ with respect to the Hilbert space topology. Clearly $\mathcal{L} = \mathcal{F}_{R,B;m}(\mathcal{H})$ satisfies all the requirements.

For type B representations there is no distinguished Hilbert space topology for completing $\mathcal{D}$. A natural substitute is the topology τ defined by the family of seminorms

$$s_\psi(\varphi) \equiv |\langle \psi\,,\,\varphi\rangle_m| \quad , \qquad \varphi\,,\psi \in \mathcal{D} \quad . \tag{3.38}$$

It turns out [2] that τ is the weakest locally convex topology in which $\langle\,\cdot\,,\,\cdot\,\rangle_m$ is separately τ-continuous. Moreover, τ is a Hausdorff topology, because $\langle\,\cdot\,,\,\cdot\,\rangle_m$ is nondegenerate. Therefore $\mathcal{D}$ admits a unique (up to isomorphism) τ-completion, which has all the needed properties and provides the space $\mathcal{L}$ for the B-series.

We conclude this section by a general observation, which concerns A-type representations only and is based on the fact that any $m \in \mathcal{M}(R,B)_+$ can be written in the form $m(x) = p^\dagger(x)\,p(x)$, where $p(x)$ is an invertible matrix. Notice that $p(x)$ is not unitary unless $m(x) = e$. It is easy to show that the mapping induced by

$$a_\alpha(x) \longmapsto p_\alpha^\beta(x)\,a_\beta(x) \quad , \qquad a^{*\alpha(x)} \longmapsto a^{*\beta}(x)\,p^{-1}{}_\beta^\alpha(x) \quad , \tag{3.39}$$

$$b_\alpha^\beta(x) \longmapsto p_\alpha^\gamma(x)\,b_\gamma^\delta(x)\,p^{-1}{}_\delta^\beta(-x) \tag{3.40}$$

is an isomorphism between $\{\mathcal{B}_R,\, I_m\}$ and $\{\mathcal{B}_{R'},\, I_e\}$, where

$$R'^{\beta_1\beta_2}_{\alpha_1\alpha_2}(x_1,x_2) = p^{\gamma_1}_{\alpha_1}(x_1)\, p^{\gamma_2}_{\alpha_2}(x_2)\, R^{\delta_1\delta_2}_{\gamma_1\gamma_2}(x_1,x_2)\, p^{-1\beta_1}_{\ \ \delta_1}(x_2)\, p^{-1\beta_2}_{\ \ \delta_2}(x_1) \quad . \tag{3.41}$$

Setting

$$B'^{\beta}_{\alpha}(x) = p^{\gamma}_{\alpha}(x)\, B^{\delta}_{\gamma}(x)\, p^{-1\beta}_{\ \ \delta}(-x) \quad , \tag{3.42}$$

one has in addition that $\mathcal{F}_{R,B;m}$ and $\mathcal{F}_{R',B';e}$ are equivalent. In other words, for any $m \in \mathcal{M}(R,B)_+$ one can equivalently replace I_m with I_e, suitably modifying (see eqs.(3.41,42)) the exchange factor R and the reflection matrix B.

Let us mention finally that the above formalism carries over easily to the Fock representations of $\mathcal{B}_{R,\lambda}$. One must only replace the Lebesgue measure $d^s x$ by the λ-invariant measure $|\mathrm{det}\lambda'(x)|^{1/2} d^s x$.

4. Applications

4.1. Free Boson Field on the Half Line

In order to give a first idea about the physical content of the algebra $\mathcal{B}_R$, we focus below on a simple example of quantization in $\mathbf{R}_+$. More precisely, we construct the free boson field $\Phi(t,x)$, satisfying

$$\left(\partial_t^2 - \partial_x^2 + M^2\right) \Phi(t,x) = 0 \quad , \qquad x \in \mathbf{R}_+ \quad , \tag{4.1}$$

with the boundary condition

$$\lim_{x \downarrow 0} (\partial_x - \eta)\, \Phi(t,x) = 0 \quad , \qquad \eta \geq 0 \quad . \tag{4.2}$$

The standard Neumann and Dirichlet boundary conditions are recovered from (4.2) by setting $\eta = 0$ or taking the limit $\eta \to \infty$ respectively.

We will show that the quantization of the system (4.1,2) can be described in terms of $\mathcal{B}_R$ with $N = 1$ and $R = 1$. The exchange structure of this boundary algebra is trivial, which allows to isolate and easily illustrate the physical implications of the boundary

generator $b(k)$. In this section the arguments of the $\mathcal{B}_R$-generators have the meaning of momenta and are denoted therefore by k, p, etc.

Let us introduce the phase factor

$$B(k) = \frac{k - i\eta}{k + i\eta} \quad . \tag{4.3}$$

Then the triplet $\{R = 1, B; m = e\}$ satisfies all requirements of the previous section and one can construct the corresponding Fock representation $\mathcal{F}_{1,B;e}$. Eq.(3.30) shows that the operator $b(k)$ acts as a multiplication by $B(k)$. Therefore, one is left in $\mathcal{F}_{1,B;e}$ with the following relations:

$$\begin{aligned} [a(k)\, ,\, a(p)] &= 0 \quad , \\ [a^*(k)\, ,\, a^*(p)] &= 0 \quad , \\ [a(k)\, ,\, a^*(p)] &= \frac{1}{2}\delta(k-p) + \frac{1}{2}B(k)\delta(k+p) \quad . \end{aligned} \tag{4.4}$$

Notice that these would be the standard canonical commutation relations, apart from the term $B(k)\delta(k+p)$. We define now the field operator

$$\Phi(t,x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi\omega(k)}} \left[a(k)\,\mathrm{e}^{-i\omega(k)t+ikx} + a^*(k)\,\mathrm{e}^{i\omega(k)t-ikx}\right] \quad , \tag{4.5}$$

where

$$\omega(k) = \sqrt{M^2 + k^2} \quad . \tag{4.6}$$

This is just the expression in the case without boundary, but one should keep in mind that now the algebra of creation and annihilation operators is different.

By means of (4.4) one easily derives the basic correlator - the two-point Wightman function

$$\langle\Omega\, ,\, \Phi(t_1,x_1)\Phi(t_2,x_2)\Omega\rangle_e = \int_{-\infty}^{\infty} \frac{dk}{4\pi\omega(k)}\mathrm{e}^{-i\omega(k)t_{12}} \left[\mathrm{e}^{-ik(x_1-x_2)} + B(k)\mathrm{e}^{-ik(x_1+x_2)}\right] , \tag{4.7}$$

where $t_{12} = t_1 - t_2$. The right hand side of eq.(4.7) defines a tempered distribution ($B(k)$ is C^∞ and bounded on $\mathbf{R}$), which satisfies eqs.(4.1,2). It consists of two terms. The term without $B(k)$ is the usual two-point Wightman function of the system without boundary. The term proportional to $B(k)$ has its origin in the boundary generator and

explicitly breaks translation and Lorentz invariance. It is remarkable that in spite of this fact, $\Phi(t,x)$ is a local field. The validity of this statement can be deduced from the commutator

$$[\,\Phi(t_1,x_1)\,,\,\Phi(t_2,x_2)\,] = iD(t_1-t_2,x_1,x_2) \quad . \tag{4.8}$$

One has

$$D(t,x_1,x_2) = \Delta(t,x_1-x_2) + \widetilde{\Delta}(t,x_1+x_2) \quad , \tag{4.9}$$

where

$$\Delta(t,x_1-x_2) = -\int_{-\infty}^{\infty} \frac{dk}{2\pi\omega(k)}\, \sin[\omega(k)t]\, \mathrm{e}^{ik(x_1-x_2)} \tag{4.10}$$

is the ordinary Pauli-Jordan function with mass M and

$$\widetilde{\Delta}(t,x_1+x_2) = -\int_{-\infty}^{\infty} \frac{dk}{2\pi\omega(k)}\, \sin[\omega(k)t]\, B(k)\, \mathrm{e}^{ik(x_1+x_2)} \quad . \tag{4.11}$$

Observing that for $x_1,\, x_2 \in \mathbf{R}_+$ the inequality $|t_1-t_2| < |x_1-x_2|$ implies $|t_1-t_2| < x_1+x_2$, one concludes that the locality properties of the field Φ are governed by the behavior of $\widetilde{\Delta}(t,x)$ for $|t|<x$. The latter can be easily evaluated and using that $\eta \geq 0$, one finds

$$\widetilde{\Delta}(t,x)\big|_{|t|<x} = 0 \quad . \tag{4.12}$$

So, $\Phi(t,x)$ is a local field when $x \in \mathbf{R}_+$. Notice that this is not the case if $\Phi(t,x)$ is considered on the whole real line. The two terms Δ and $\widetilde{\Delta}$ in the commutator have a very intuitive explanation. As far as $|t_1-t_2| < |x_1-x_2|$ no signal can propagate between the points (t_1,x_1) and (t_2,x_2) and the commutator vanishes. When $|x_1-x_2| < |t_1-t_2| < x_1+x_2$ signals can propagate directly between the two points, but they cannot be influenced by the boundary and the only contribution comes from the standard Pauli-Jordan function Δ. As soon as $x_1+x_2 = |t_1-t_2|$, signals starting from one of the points can be reflected at the boundary and reach the other point. This phenomenon is responsible for the term $\widetilde{\Delta}$, and is codified in term proportional to $B(k)$ of the boundary algebra (4.4).

The case $\eta < 0$ is slightly more delicate due to the presence of a bound state in the one-particle energy spectrum, which must be taken into account in the construction of a local field.

The results of this subsection can be obviously generalized to higher space-time dimensions.

4.2. Scattering on the Half Line

Before entering the details of the application of $\mathcal{B}_R$ to factorized scattering with reflecting boundary conditions, we will discuss the simple case of particles of mass M freely moving on $\mathbf{R}_+$ and bouncing over a wall at $x = 0$. The relevant one-particle space is $L^2(\mathbf{R}_+, dx)$. We denote by $D_\eta \subset L^2(\mathbf{R}_+, dx)$ the subspace of C^∞-functions on $\mathbf{R}_+$, which vanish for sufficiently large x, have square integrable first and second derivatives and obey

$$\lim_{x\downarrow 0} \left(\frac{d}{dx} - \eta\right)\varphi(x) = 0 \quad . \tag{4.13}$$

The current

$$j = -\frac{i}{2m}\left[\overline{\varphi}\frac{d\varphi}{dx} - \frac{d\,\overline{\varphi}}{dx}\varphi\right] \tag{4.14}$$

satisfies $j(0) = 0$ for all $\varphi \in D_\eta$, thus preventing any probability flow through the wall $x = 0$. For one-particle Hamiltonian we take

$$H^{(1)} = -\frac{1}{2M}\triangle \quad , \tag{4.15}$$

defined on D_η. The evolution problem is well posed because $H^{(1)}$, which is obviously symmetric, is actually essentially self-adjoint [22]. A set of (generalized) eigenstates verifying (4.13) is

$$\psi_k(x) = \mathrm{e}^{-ikx} + B(k)\mathrm{e}^{ikx} \quad , \qquad k \in \mathbf{R} \quad , \tag{4.16}$$

where $B(k)$ is given by eq.(4.3). The eigenvectors (4.16), which represent physically scattering states, satisfy

$$\psi_{-k}(x) = \overline{\psi}_k(x) = B(-k)\psi_k(x) \quad . \tag{4.17}$$

For $\eta \geq 0$ the systems $\{\psi_k \,:\, k > 0\}$ and $\{\psi_{-k} \,:\, k > 0\}$ are separately complete and are related via complex conjugation, which in the physical context implements time

reversal. When $\eta < 0$, there is in addition a unique bound state

$$\psi_{\rm b}(x) = \sqrt{-2\eta}\, {\rm e}^{\eta x} \quad , \tag{4.18}$$

with energy $E = -\eta^2/2M$.

The n-body Hamiltonian of the associated multiparticle Bose system

$$H^{(n)} = -\frac{1}{2M}(\triangle_1 + ... + \triangle_n) \tag{4.19}$$

is defined on $D^n_{\eta +}$ - the subspace of symmetric functions in $D_\eta^{\otimes n}$. Clearly, there is neither particle production nor particle collision in this model. There is however a nontrivial reflection from the boundary, which can be described as follows. One can consider ψ_k as representing a particle, which when time $t \to -\infty$, travels with momentum $-k$ towards the wall. Accordingly, we take

$$|-k\rangle^{\rm in} = \frac{1}{\sqrt{2\pi}}\, \psi_k(x) \quad , \qquad k > 0 \quad , \tag{4.20}$$

as a basis of one-particle "in"-states. Concerning the basis of one-particle "out"-states, the analogous consideration gives

$$|k\rangle^{\rm out} = \frac{1}{\sqrt{2\pi}}\, \overline{\psi}_k(x) = \frac{1}{\sqrt{2\pi}}\, \psi_{-k}(x) \quad , \qquad k > 0 \quad . \tag{4.21}$$

The scattering operator is defined at this point by

$$S\, |k\rangle^{\rm out} = |-k\rangle^{\rm in} \quad . \tag{4.22}$$

For $\eta \geq 0$, S is by construction a unitary operator on $L^2(\mathbf{R}_+, dx)$. For $\eta < 0$, S is defined and unitary on the subspace of $L^2(\mathbf{R}_+, dx)$ which is orthogonal to the bound state (4.18). The one-particle matrix elements of S read

$${}^{\rm out}\langle k|S|p\rangle^{\rm out} = {}^{\rm out}\langle k|-p\rangle^{\rm in} = \frac{1}{2\pi}\int_0^\infty dx \psi_k(x)\psi_p(x) = B(k)\delta(k-p) \quad . \tag{4.23}$$

More generally

$${}^{\rm out}\langle k_1, ..., k_n| - p_1, ..., -p_n\rangle^{\rm in} = B(k_1)...B(k_n)\delta(k_1 - p_1)...\delta(k_n - p_n) \quad , \tag{4.24}$$

provided that $k_1 > ... > k_n > 0$ and $p_1 > ... > p_n > 0$.

Our main observation now is that the above simple scattering problem admits a field-theoretic solution in terms of the algebra (4.4). In fact, it is easy to verify that the vacuum expectation values

$$2^n \langle a^*(k_1)...a^*(k_n)\Omega \, , \, a^*(-p_1)...a^*(-p_n)\Omega\rangle_e \quad , \tag{4.25}$$

in the Fock representation $\mathcal{F}_{1,B;e}$ reproduce precisely the transition amplitudes (4.24). We have therefore the following Fock realization

$$|k_1, ..., k_n\rangle^{\rm out} = 2^{\frac{n}{2}} a^*(k_1)...a^*(k_n)\Omega \quad , \qquad k_1 > ... > k_n > 0 \quad , \tag{4.26}$$

$$| - p_1, ..., -p_n\rangle^{\rm in} = 2^{\frac{n}{2}} a^*(-p_1)...a^*(-p_n)\Omega \quad , \qquad p_1 > ... > p_n > 0 \quad , \tag{4.27}$$

of the interpolating states. Summarizing, the scattering operator of our simple model has a purely algebraic characterization. In this respect, the term proportional to $B(k)$ in (4.4) is the algebraic counterpart of the boundary condition, given analytically by eq.(4.13).

At this stage we have enough background for facing the more complicated problem of scattering in integrable models with reflecting boundary conditions in 1+1 space-time dimensions. The presence of particle collisions in this case leads in general to the boundary algebras $\mathcal{B}_R$ with $R \neq 1$. Using the Fock representations of $\mathcal{B}_R$, derived in the previous section, we present below a rigorous construction of the S-matrix, which generalizes some previous results [20] valid in the absence of a boundary. We also show that under certain conditions on the triplet $\{R, B; m\}$, the transition amplitudes, originally derived by Cherednik [4], are indeed Hilbert space matrix elements of a unitary operator.

The asymptotic particles of integrable models are parametrized by their rapidity $\theta \in \mathbf{R}$ and internal "isotopic" index $\alpha = 1, ..., N$. We recall that in the case of relativistic dispersion relation the energy-momentum vector is expressed in terms of θ and the mass M according to

$$p^0 = M \cosh(\theta) \ , \qquad p^1 = M \sinh(\theta) \ . \tag{4.28}$$

An elastic reflection $(p^0,\, p^1) \longmapsto (p^0,\, -p^1)$ corresponds therefore to the transformation $\theta \longmapsto -\theta$.

The fundamental building blocks for constructing the scattering operator are the matrices $R_{\alpha_1\alpha_2}^{\beta_1\beta_2}(\theta_1,\theta_2)$ and $B_\alpha^\beta(\theta)$, which are supposed to satisfy eqs.(2.9,10) and (3.9,10). We allow for R to depend on θ_1 and θ_2 separately (and not only on $\theta_1 - \theta_2$), because in general the presence of boundaries brakes down Lorentz invariance.

A crucial observation is that the algebra $\mathcal{B}_R$ alone does not determine the scattering operator S we are looking for: one must fix in addition an involution I_m. The latter selects a Fock representation $\mathcal{F}_{R,B;m}$, which is the main ingredient for constructing S. Postponing the discussion of the physical meaning of the choice of $m \in \mathcal{M}(R,B)$ to the end of this section, it might be instructive for the time being to describe the set $\mathcal{M}(R,B)$ for some familiar integrable model. We choose the $SU(2)$ Thirring model. In this case $N = 2$ and setting $\theta_{12} = \theta_1 - \theta_2$ the relevant R-matrix reads [1]

$$R(\theta_1,\theta_2) = \frac{i\pi\rho(\theta_{12})}{(i\pi - \theta_{12})\rho(-\theta_{12})} \sum_{\alpha,\beta=1}^{2} \left[E_{\alpha\alpha} \otimes E_{\beta\beta} + \frac{\theta_{12}}{i\pi}(-1)^{\alpha+\beta} E_{\alpha\beta} \otimes E_{\beta\alpha}\right] , \quad (4.29)$$

where $E_{\alpha\beta}$ are the Weyl matrices and

$$\rho(\theta) = \Gamma\left(\frac{1}{2} + \frac{\theta}{2\pi i}\right) \Gamma\left(1 - \frac{\theta}{2\pi i}\right) \quad . \quad (4.30)$$

The general solution of eqs.(3.9,10), subject to the physical constraint of boundary crossing symmetry [10], is given in [3]. Let us concentrate for simplicity on the diagonal solutions

$$B(\theta) = \frac{\beta(\theta)}{\beta(-\theta)} \left(E_{11} + \frac{\eta - \theta}{\eta + \theta} E_{22}\right) \quad , \quad (4.31)$$

with $\eta \in \mathbf{C}$ and

$$\beta(\theta) = \Gamma\left(\frac{3}{4} + \frac{\theta}{2\pi i}\right) \Gamma\left(1 - \frac{\theta}{2\pi i}\right) \Gamma\left(\frac{\eta + i\pi - \theta}{2\pi i}\right) \Gamma\left(\frac{\eta + 2\pi i + \theta}{2\pi i}\right) \quad . \quad (4.32)$$

Let μ_+ (μ_-) be any measurable real-valued even (odd) function, such that $\mu_\pm$ and $1/\mu_\pm$ are bounded. Then, if $\mathrm{Re}\,\eta = 0$, the set $\mathcal{M}(R,B)$ contains all matrices of the form

$$m(\theta) = \mu_+(\theta)\,(E_{11} + \xi E_{22}) \quad , \qquad \xi \in \mathbf{R},\ \xi \neq 0 \quad . \quad (4.33)$$

In addition, for $\eta = 0$ one has the solutions

$$m(\theta) = \mu_-(\theta)\left(\zeta E_{12} + \bar{\zeta} E_{21}\right) \quad , \qquad \zeta \in \mathbf{C} \quad . \tag{4.34}$$

From eq.(4.33) it follows that $\mathcal{M}(R,B)_+ \neq \emptyset$.

After this concrete example illustrating the set $\mathcal{M}(R,B)$, we return to the general framework. The idea is to extend the formalism, developed at the beginning of this section for the Schrödinger particle on the half line, to the case of integrable models. In what follows we assume that

$$\mathcal{M}(R,B)_+ \neq \emptyset \tag{4.35}$$

and consider representations $\mathcal{F}_{R,B;m}$ of type A. The physical motivation for this restriction is quite evident. According to proposition 7, it ensures positivity of the metric in the asymptotic spaces $\mathcal{F}^{\rm out}$ and $\mathcal{F}^{\rm in}$, which we are going to construct now. For this purpose we introduce the following relation in $C_0^\infty(\mathbf{R})$:

$$f_1 \succ f_2 \quad \Longleftrightarrow \quad \theta_1 > \theta_2 \quad \forall\, \theta_1 \in {\rm supp}(f_1)\, , \quad \forall\, \theta_2 \in {\rm supp}(f_2) \quad . \tag{4.36}$$

We will adopt also the notation

$$f \succ 0 \quad \Longleftrightarrow \quad \theta > 0 \quad \forall\, \theta \in {\rm supp}(f) \quad , \tag{4.37}$$

and

$$\widetilde{f}(\theta) = f(-\theta) \quad . \tag{4.38}$$

As suggested by eqs.(4.26,27), $\mathcal{F}^{\rm out}$ and $\mathcal{F}^{\rm in}$ are generated by finite linear combinations of the vectors ($k \geq 1$)

$$\mathcal{E}^{\rm out} = \{\, \Omega,\, a^*(f_1)\cdots a^*(f_k)\,\Omega \ :\ f_{1_{\alpha_1}} \succ \cdots \succ f_{k_{\alpha_k}} \succ 0,\, \forall\, \alpha_1, ..., \alpha_k = 1, ..., N \,\} \tag{4.39}$$

and

$$\mathcal{E}^{\rm in} = \{\, \Omega,\, a^*(\widetilde{g}_1)\cdots a^*(\widetilde{g}_k)\,\Omega \ :\ g_{1_{\beta_1}} \succ \cdots \succ g_{k_{\beta_k}} \succ 0,\, \forall\, \beta_1, ..., \beta_k = 1, ..., N \,\} \tag{4.40}$$

respectively. By construction both $\mathcal{F}^{\rm out}$ and $\mathcal{F}^{\rm in}$ are linear subspaces of the Hilbert space $\mathcal{F}_{R,B;m}(\mathcal{H})$.

One should notice that in principle there are elements of $\mathcal{F}^0_{R,B;m}(\mathcal{H})$ which belong neither to $\mathcal{F}^{\rm out}$ nor to $\mathcal{F}^{\rm in}$. We call them mixed vectors. Linear combinations involving both in- and out-states provide in general examples of such vectors. In spite of the existence of mixed vectors, the subspaces $\mathcal{F}^{\rm out}$ and $\mathcal{F}^{\rm in}$ satisfy a sort of asymptotic completeness, which is essential for constructing the S-matrix. More precisely, one has

Proposition 8. *$\mathcal{F}^{\rm out}$ and $\mathcal{F}^{\rm in}$ separately are dense in $\mathcal{F}_{R,B;m}(\mathcal{H})$.*

Proof: We focus on $\mathcal{F}^{\rm out}$. Let $\varphi \in \mathcal{F}_{R,B;m}(\mathcal{H})$ and let us assume that

$$\langle \varphi \, , \, \psi \rangle_m = 0 \quad \forall \, \psi \in \mathcal{F}^{\rm out} \quad . \tag{4.41}$$

In order to prove the thesis, we have to show that $\varphi = \left(\varphi^{(0)}, \varphi^{(1)}, ..., \varphi^{(n)}, ...\right) = 0$. Obviously $\varphi^{(0)} = 0$. Let us consider $\varphi^{(n)}$ for arbitrary but fixed $n \geq 1$. Eq.(3.27) and eq.(4.41) imply that

$$\langle \varphi^{(n)} \, , \, a^*(f_1) \cdots a^*(f_n) \Omega \rangle_m =$$
$$\int d\theta_1 \cdots d\theta_n \, \varphi^{(n)\dagger \alpha_1 \ldots \alpha_n}(\theta_1, ..., \theta_n) m^{\beta_1}_{\alpha_1}(\theta_1) \cdots m^{\beta_n}_{\alpha_n}(\theta_n) f_{1_{\beta_1}}(\theta_1) \cdots f_{n_{\beta_n}}(\theta_n) = 0 \tag{4.42}$$

for all $f_1, ..., f_n$ such that $f_{1_{\alpha_1}} \succ \cdots \succ f_{n_{\alpha_n}} \succ 0 \; \forall \, \alpha_1, ..., \alpha_n = 1, ..., N$. Therefore

$$\varphi^{(n)}_{\alpha_1 \ldots \alpha_n}(\theta_1, ..., \theta_n) = 0 \tag{4.43}$$

in the domain $\theta_1 > \cdots > \theta_n > 0$. Finally, using that $\varphi^{(n)} \in \mathcal{H}^n_{R,B}$ has definite exchange and reflection properties described by eqs.(3.24,25), one can extend the domain of validity of (4.43) and conclude that $\varphi^{(n)}$ actually vanishes almost everywhere in $\mathbf{R}^n$. Clearly, a similar argument applies also to the case of $\mathcal{F}^{\rm in}$.

We observe in passing that the definition of $\mathcal{F}^{\rm out}$ and $\mathcal{F}^{\rm in}$ does not explicitly involve the boundary generators $\{b^\beta_\alpha(\theta)\}$. This fact is not surprising because Ω is cyclic with respect to $\{a^{*\alpha}(\theta)\}$.

At this point we are ready to define the scattering matrix S and to prove that it is a unitary operator in $\mathcal{F}_{R,B;m}(\mathcal{H})$. The construction consists essentially of three steps. One starts by defining S as the following mapping of $\mathcal{E}^{\rm out}$ onto $\mathcal{E}^{\rm in}$:

$$S\Omega = \Omega \quad , \tag{4.44}$$

$$S\, a^*(g_1)a^*(g_2)\cdots a^*(g_k)\Omega = a^*(\widetilde{g}_1)a^*(\widetilde{g}_2)\cdots a^*(\widetilde{g}_k)\Omega \quad , \tag{4.45}$$

where $g_{1_{\beta_1}} \succ \cdots \succ g_{k_{\beta_k}} \succ 0,\ \forall\, \beta_1, ..., \beta_k = 1, ..., N$. It is not difficult to check that

$$\langle S\psi^{\rm out}\, ,\, S\varphi^{\rm out}\rangle_m \;=\; \langle \psi^{\rm out}\, ,\, \varphi^{\rm out}\rangle_m \quad , \qquad \forall\, \psi^{\rm out},\, \varphi^{\rm out} \in \mathcal{E}^{\rm out} \quad . \tag{4.46}$$

Moreover, S is invertible and

$$\langle S^{-1}\psi^{\rm in}\, ,\, S^{-1}\varphi^{\rm in}\rangle_m \;=\; \langle \psi^{\rm in}\, ,\, \varphi^{\rm in}\rangle_m \quad , \qquad \forall\, \psi^{\rm in},\, \varphi^{\rm in} \in \mathcal{E}^{\rm in} \quad . \tag{4.47}$$

The second step is to extend S and S^{-1} by linearity to the whole $\mathcal{F}^{\rm out}$ and $\mathcal{F}^{\rm in}$ respectively. Clearly, one has to show that these extensions are correctly defined. Consider for instance S and suppose that there exist a sequence

$$g^i_{1_{\beta_1}} \succ \cdots \succ g^i_{k_{\beta_k}} \succ 0\, , \quad \forall\, \beta_1, ..., \beta_k = 1, ..., N\, , \quad i = 1, ..., M\, ,$$

such that

$$a^*(g_1)a^*(g_2)\cdots a^*(g_k)\Omega = \sum_{i=1}^{M} a^*(g^i_1)a^*(g^i_2)\cdots a^*(g^i_k)\Omega \quad . \tag{4.48}$$

In order to prove that the linear extension of S is not ambiguous, we must show that

$$a^*(\widetilde{g}_1)a^*(\widetilde{g}_2)\cdots a^*(\widetilde{g}_k)\Omega = \sum_{i=1}^{M} a^*(\widetilde{g}^i_1)a^*(\widetilde{g}^i_2)\cdots a^*(\widetilde{g}^i_k)\Omega \quad . \tag{4.49}$$

The argument is as follows. In the domain $\theta_1 > \theta_2 > ... > \theta_k > 0$ eq.(4.48) implies that

$$g_{1_{\beta_1}}(\theta_1)\, g_{2_{\beta_2}}(\theta_2)\cdots g_{k_{\beta_k}}(\theta_k) = \sum_{i=1}^{M} g^i_{1_{\beta_1}}(\theta_1)\, g^i_{2_{\beta_2}}(\theta_2)\cdots g^i_{k_{\beta_k}}(\theta_k) \quad . \tag{4.50}$$

Because of the support properties of $\{g_j\}$ and $\{g^i_j\}$ one has that eq.(4.50) holds actually in $\mathbf{R}^k$, which projected by $P^{(k)}_{R,B}$ proves the validity of eq.(4.49).

It is easy to see also that eqs.(4.46,47) remain valid for the linear extensions of S and S^{-1} on $\mathcal{F}^{\rm out}$ and $\mathcal{F}^{\rm in}$ respectively. This fact implies in particular that both S and S^{-1} are bounded linear operators.

Finally, one extends S and S^{-1} by continuity to $\mathcal{F}_{R,B;m}(\mathcal{H})$. Because of the asymptotic completeness proven in proposition 8, the extensions are unique and define the

unitary scattering operator and its inverse. As it should be expected from integrability, one has $S\mathcal{H}^n_{R,B} \subset \mathcal{H}^n_{R,B}$. Notice however, that in contrast to the case without boundary, where the scattering operator leaves invariant each one-particle state, the S-matrix constructed above acts nontrivialy already in $\mathcal{H}^1_{R,B}$.

By construction the matrix elements of S between out-states in the Fock space $\mathcal{F}_{R,B;e}(\mathcal{H})$ reproduce precisely the transition amplitudes derived by Cherednik [4]. Since the latter are referred to the involution I_e, a natural question arising at this point concerns the physical meaning of other possible choices of $m \in \mathcal{M}(R,B)_+$. For answering this question we consider two generic asymptotic states $\varphi^{\rm in} \in \mathcal{F}^{\rm in}$ and $\psi^{\rm out} \in \mathcal{F}^{\rm out}$. If both m, $e \in \mathcal{M}(R,B)_+$, one may compare the transition amplitudes associated with the involutions I_m and I_e. One finds

$$\langle \psi^{\rm out}\, ,\, \varphi^{\rm in} \rangle_m \, = \, \langle \psi^{\rm out}\, ,\, \varphi^{\rm in}_d \rangle_e \, = \, \langle \psi^{\rm out}_d\, ,\, \varphi^{\rm in} \rangle_e \quad , \tag{4.51}$$

where $\varphi^{\rm in}_d$ and $\psi^{\rm out}_d$ are the "dressed" in- and out-states

$$(\varphi^{\rm in}_d)^{(n)}_{\alpha_1\ldots\alpha_n}(\theta_1,...,\theta_n) = m_{\alpha_1}^{\gamma_1}(\theta_1)\cdots m_{\alpha_n}^{\gamma_n}(\theta_n)(\varphi^{\rm in})^{(n)}_{\gamma_1\ldots\gamma_n}(\theta_1,...,\theta_n) \quad , \tag{4.52}$$

$$(\psi^{\rm out}_d)^{(n)}_{\beta_1\ldots\beta_n}(\theta_1,...,\theta_n) = m^{\dagger\gamma_1}_{\ \beta_1}(\theta_1)\cdots m^{\dagger\gamma_n}_{\ \beta_n}(\theta_n)(\psi^{\rm out})^{(n)}_{\gamma_1\ldots\gamma_n}(\theta_1,...,\theta_n) \quad . \tag{4.53}$$

It follows from eq.(4.51) that the effect of the involution I_m is exactly reproduced in $\mathcal{F}_{R,B;e}$ by appropriate dressing (4.52,53) of the in- or out-states.

The results of this section can be summarized as follows.

Proposition 9. *Suppose that the exchange factor R and the reflection matrix B satisfy* (2.9,10) *and* (3.9,10). *Assume also that $\mathcal{M}(R,B)_+ \neq \emptyset$. Then the scattering operator associated with the Fock representation $\mathcal{F}_{R,B;m}$ is unitary for any $m \in \mathcal{M}(R,B)_+$.*

Conditions (2.9,10) and (3.9,10) are standard for the scattering on the half line. The same is true for (2.15), which is usually imposed in the slightly stronger form

$$R^{\dagger\beta_1\beta_2}_{\ \alpha_1\alpha_2}(\theta_1,\theta_2) = R_{\alpha_1\alpha_2}^{\beta_1\beta_2}(\theta_2,\theta_1) \quad , \tag{4.54}$$

known as Hermitian analyticity. We emphasize that condition (3.6), which is often overlooked in the physical literature, is essential for the unitarity of S and represents

therefore an useful criterion for selecting possible reflection matrices. In the case of the $SU(2)$ Thirring model one gets in this way the restriction $\mathrm{Re}\,\eta = 0$ in eqs.(4.31,32).

Let us mention also that if R depends on the difference $\theta_{12} \equiv \theta_1 - \theta_2$, one usually assumes [13,26] that R admits a suitable continuation to the complex θ_{12}-plane, which satisfies crossing symmetry, has certain pole structure, etc. In that case also B is required to have a continuation in the complex θ-plane, which obeys boundary crossing symmetry [10]. In our example (see eqs.(4.29-32)) R and B admit such continuations. Finally, the bootstrap equations [9,26] reduce further the set of physically relevant exchange and reflection matrices. From proposition 9 it follows however that the unitarity of S as an operator in $\mathcal{F}_{R,B;m}(\mathcal{H})$ depends exclusively on the behavior of R and B for real values of the rapidities.

5. Outlook and Conclusions

In the present paper we have introduced the associative algebra $\mathcal{B}_R$ and investigated some of its basic features. $\mathcal{B}_R$ admits two series of Fock representations, which have been constructed explicitly. The positive metric representations provide a framework for deriving Cherednik's transition amplitudes and proving that they are indeed the matrix elements of a uniptary scattering operator. We have shown also that the algebra $\mathcal{B}_+$ enters the Bose quantization on the half line. The associated Klein-Gordon field is local, in spite the breakdown of the Poincaré symmetry.

$\mathcal{B}_R$ is actually a member of a large family of algebras $\mathcal{B}_{R,\lambda}$, which are defined by eqs.(2.23-26). $\mathcal{B}_{R,\lambda}$ can be studied in the same way as $\mathcal{B}_R$ and are expected to find relevant applications to statistical models with boundaries. It will be interesting in this respect to extend to $\mathcal{B}_{R,\lambda}$ the notion of second R-quantization, developed in [17,18] for the Z-F algebra $\mathcal{A}_R$.

We point out finally that one can further generalize $\mathcal{B}_{R,\lambda}$, eliminating the condition (2.9) and/or (3.10). In this case, instead with the Weyl group $\mathcal{W}_n$, one has to deal with an infinite dimensional group $\mathcal{W}'_n$, which is freely generated by the elements $\{\tau', \sigma'_i \; : \; i = 1, ..., n-1\}$ satisfying the relations (3.14,15), but not (3.16). Recent

investigations [15] show actually that the group $\mathcal{W}_n'$ appears in many different physical and mathematical contexts. We hope to say more about this generalization of $\mathcal{B}_{R,\lambda}$ in the near future.

Appendix

In quantum field theory on the half line it is sometimes necessary to allow for a quantum number $j = 1, ..., N_B$ to reside on the boundary [10]. We will show below that this case is still described by the boundary algebra $\{\mathcal{B}_R, I_m\}$, but corresponds to representations with slightly more general structure then that of $\mathcal{F}_{R,B;m}$. To be precise, instead of the requirement 4 formulated in the beginning of Sect. 3, these representations satisfy:

$4'$. There exists a N_B-dimensional subspace (vacuum space) $\mathcal{V} \subset \mathcal{D}$, which is annihilated by $a_\alpha(x)$. Moreover, $\mathcal{V}$ is cyclic with respect to $\{a^{*\alpha}(x)\}$ and $\langle \cdot\,, \cdot \rangle_m$ is positive definite on $\mathcal{V}$.

For $N_B = 1$ we recover the property 4 specifying $\mathcal{F}_{R,B;m}$.

Let us briefly describe now the main features of the representations characterized by the conditions 1-3 and $4'$. Let $\Omega^1, \ldots, \Omega^{N_B}$ be an orthonormal basis in $\mathcal{V}$. We denote by P_0 be the $\langle \cdot\,, \cdot \rangle_m$-orthogonal projection on $\mathcal{V}$ and define

$$B_\alpha^\beta(x) \equiv P_0\, b_\alpha^\beta(x)\, P_0 \quad . \tag{A.1}$$

Notice that $B_\alpha^\beta(x)$ is now an operator, carrying the vacuum space into itself,

$$B_\alpha^\beta(x)\, \Omega^j = B_{\alpha\ k}^{\beta\ j}(x)\, \Omega^k \quad . \tag{A.2}$$

The following obvious generalization of Proposition 3 holds.

Proposition $3'$. *The vacuum space $\mathcal{V}$ is unique and satisfies*

$$b_\alpha^\beta(x)\,|_\mathcal{V} = B_\alpha^\beta(x)\,|_\mathcal{V} \quad . \tag{A.3}$$

Projecting the relevant equations on the vacuum space, one immediately verifies the validity of (3.6,9,10) as operator equations on $\mathcal{V}$.

Summarizing, the basic input for constructing the above more general class of representations of $\{\mathcal{B}_R, I_m\}$ is still the triplet $\{R, B; m\}$, the novelty being that $B_\alpha^\beta(x)$ are operators which satisfy (3.6,9,10) on $\mathcal{V}$. Apart from the following minor modifications, the construction precisely follows that described in Sect. 3. First of all, the elements of $\mathcal{H}^n_{R,B}$ carry an extra lower index varying from 1 to N_B. In the scalar product this index is saturated among the two states. Second, performing the substitution $B_{\alpha_i}^{\beta_i}(x) \mapsto B_{\alpha_i}^{\beta_i}{}^j_k(x)$ in eq.(3.20), the operator $B_i(x)$ becomes a $N_B \times N_B$-matrix, which inserted in (3.30,31) acts on the states by a standard matrix multiplication.

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